

Completing the Square — Prepared by Nikolas Agisilaou

What Is Completing the Square?

Rewrite $ax^2 + bx + c$ as $a(x + p)^2 + q$.

- **Core trick:** constant = $\left(\frac{\text{coeff of } x}{2}\right)^2$
- Reveals solutions, vertex, min/max.
- Works on *any* quadratic.
- $y = a(x - h)^2 + k$: vertex (h, k) ;
 $a > 0$: minimum $a < 0$: maximum

Perfect Square Identities — The Foundation

$$\begin{aligned} (x + a)^2 &= x^2 + 2ax + a^2 & (x - a)^2 &= x^2 - 2ax + a^2 \\ (x + 7)^2 &= x^2 + 14x + 49 & \left(\frac{14}{2}\right)^2 &= 49 & (x - 6)^2 &= x^2 - 12x + 36 \\ (x + \frac{3}{2})^2 &= x^2 + 3x + \frac{9}{4} & \left(\frac{3}{2}\right)^2 &= \frac{9}{4} & \text{Bold constant is always } \left(\frac{b}{2}\right)^2 \end{aligned}$$

The Two Methods: Step-by-Step

Case 1 — $a = 1$: $x^2 + bx + c$

1. Half the x -coefficient: $p = \frac{b}{2}$
2. $(x + p)^2$ expands to $x^2 + bx + p^2$
3. Subtract p^2 , keep c : $(x + p)^2 - p^2 + c$
4. Simplify: $q = c - p^2$

Formula: $x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$

Case 2 — $a \neq 1$: $ax^2 + bx + c$

1. Factor a from x -terms *only*: $a\left(x^2 + \frac{b}{a}x\right) + c$
2. Complete the square inside the bracket
3. Multiply a back through the bracket
4. Simplify the constants

Formula: $ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$

Worked Examples at a Glance

Expression	Key step	Result	Expression	Key step	Result
$x^2 + 10x + 3$	$p = 5, p^2 = 25$	$(x + 5)^2 - 22$	$2x^2 + 16x + 3$	$2(x^2 + 8x); 2 \times 16 = 32$	$2(x + 4)^2 - 29$
$x^2 - 12x + 11$	$p = -6, p^2 = 36$	$(x - 6)^2 - 25$	$4x^2 - 24x + 11$	$4(x^2 - 6x); 4 \times 9 = 36$	$4(x - 3)^2 - 25$
$x^2 + 9x + 4$	$p = \frac{9}{2}, p^2 = \frac{81}{4}$	$(x + \frac{9}{2})^2 - \frac{65}{4}$	$-x^2 + 8x + 3$	$-(x^2 - 8x); -(-16) = +16$	$-(x - 4)^2 + 19$

Solving Equations by Completing the Square

Steps: complete the square $\rightarrow$ rearrange $\rightarrow$ take $\pm\sqrt{}$
 $(x + p)^2 = k \implies x = -p \pm \sqrt{k}$

$$\begin{aligned} x^2 + 10x + 3 &= 0 & (x + 5)^2 &= 22 \implies x = -5 \pm \sqrt{22} \\ 2x^2 - 12x + 7 &= 0 & 2(x - 3)^2 &= 11 \implies x = 3 \pm \frac{\sqrt{22}}{2} \\ x^2 + 6x + 15 &= 0 & (x + 3)^2 &= -6 \implies \text{no real solutions} \end{aligned}$$

Number of Solutions & Discriminant Link

After completing the square: $(x + p)^2 = k$

k	Solutions	Graph
$k > 0$	Two distinct real roots	Cuts x -axis
$k = 0$	One repeated root	Touches x -axis
$k < 0$	No real roots	Misses x -axis

Same rules with $\Delta = b^2 - 4ac$: $\Delta > 0, = 0, < 0$.

Deriving the Quadratic Formula — Completing the Square on $ax^2 + bx + c$

Step 1: Divide by a : $x^2 + \frac{b}{a}x = -\frac{c}{a}$ **Step 3:** $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$
Step 2: Add $\left(\frac{b}{2a}\right)^2$ to both sides **Step 4:** $x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Common Traps & Pitfalls

1. **Subtract Trap:** $(x + 5)^2 + 3 \neq x^2 + 10x + 3$. Always subtract p^2 .
2. **Sign Trap:** Half of -12 is $-6 \implies (x - 6)^2$, not $(x + 6)^2$.
3. **Factor Trap:** Factor a from x -terms *only* — not the constant.
4. **Negative a Trap:** $-[(x - 4)^2 - 16] = -(x - 4)^2 + 16$.
5. $\pm$ **Trap:** Never drop the $\pm$ when taking a square root to solve.

Quick Exam Shortcuts

- “**Form** $(x + a)^2 + b$ ” $\rightarrow$ Complete the square
“**Surd-form answers**” $\rightarrow$ Complete the square
“**No real solutions**” $\rightarrow$ Show $(x + p)^2 = k < 0$
“**Derive quad. formula**” $\rightarrow$ CTS on $ax^2 + bx + c = 0$
Always: expand back to verify the original.
Odd- b values of p^2 : $\frac{1}{4}, \frac{9}{4}, \frac{25}{4}, \frac{49}{4}, \frac{81}{4} \dots$